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On highly convergent 2D acoustic and elastic wave propagation models

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SUMMARY

The presented approach for reducing the phase and group errors in short wavelength pulses propagation modelling is based upon modal error minimization. A computational model is built of component substructures (CS) the matrices of which are obtained by modal synthesis. The necessary modal properties of CS are established by solving the cumulative modal error minimization problem for a sample domain the exact modal frequencies of which are known theoretically. Earlier the approach has been demonstrated to work well in 1D case. In this work the results for 2D rectangular meshes describing elastic and/or acoustic wave propagation have been obtained. As a result, models having up to 80% of modal frequencies with an error less than 2% can be obtained by using the optimized component substructures. Though the synthesized mass matrices are non-diagonal, the obtained dynamic models are able to simulate short transient waves and wave pulses propagating in elastic or acoustic environments by using only a few nodal points per pulse length. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: modal synthesis; modal error; transient wave simulation

1. INTRODUCTION

The shape of a propagating short wavelength pulse simulated in a discrete mesh is inevitably distorted if the distance travelled by the pulse comprises a large number of lengths of the pulse. As a result, the shape and duration of the simulated pulse become very different from the values expected theoretically. An important source of distortions is the phase error inherently produced by the discrete model. The errors can be minimized by means of very dense meshing, however, this makes the simulation complex and requiring huge computational resources. Modal errors of a computational model can be regarded as an origin of phase errors.

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As a consequence, different harmonic components of waves comprising the wave pulse propagate with different velocities and produce group errors of wave propagation.

As early as in 1982 different modal frequency convergence features of dynamic models obtained by using lumped and consistent forms of mass matrices have been noticed [1]. The convergence of modal frequencies of dynamic models can be improved by using the 'generalized' form of the mass matrix obtained as a weighted superposition of lumped and consistent mass matrices [2]. Approaches concentrating on improvement of modal convergence properties and retaining the diagonal form of the mass matrix have been presented in References [3–6].

Element type 99 of LSDYNA program is intended for vibration studies carried out in time domain. These models may have very large numbers of elements and may be run for relatively long durations. This is achieved by imposing strict limitations on the range of applicability, thereby simplifying the calculations: elements must be cuboid; small displacement, small strain, negligible rigid body rotation; elastic material only. The element formulation also includes single element bending and torsion modes [7].

The non-diagonal matrices obtained by modal synthesis can give better results. In 1D case they produce models having 60–80% of modal frequencies with error values less than 3% [8]. In this work we demonstrate that the main principles of the approach presented in Reference [8] can be also applied for 2D rectangular wave propagation models. CS are 'optimized' in order to provide minimum cumulative modal frequency errors of selected sample domains. We demonstrate that the structure of any size assembled of such CS has approximately the same percentage of 'close-to-exact' modal frequencies. Though the synthesized mass matrices are non-diagonal, the obtained dynamic models are able to simulate transient waves by using only a few nodal points per pulse length.

2. GENERAL RELATIONS OF MODAL SYNTHESIS

Finite element models of small vibrations and waves in elastic or acoustic continua are presented by the well-known semi-discrete structural dynamic equation as

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\} + [\mathbf{C}]\{\dot{\mathbf{U}}\} + [\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{R}(t)\} \tag{1}$$

where $[\mathbf{M}], [\mathbf{K}]$ are structural mass and stiffness matrices of the element, $\{\mathbf{R}\}$ is the nodal vector containing the lumped forces. In problems addressed in this work we assume the damping forces to be very small. If necessary, slight damping can be conveniently introduced by means of the proportional damping matrix $[\mathbf{C}] = \alpha[\mathbf{M}] + \beta[\mathbf{K}]$. The reason for such a simplification is that in many ultrasonic measurement applications propagating pulses do not fade perceptibly after travelling distances considered by simulations, as well as, the investigations are mainly focused on the wave type transformations caused by reflections and interactions.

The structural matrices used in (1) can be expressed by using modal synthesis relations as

$$[\mathbf{M}] = ([\mathbf{Y}]^{\mathrm{T}})^{-1}[\mathbf{Y}]^{-1}; \quad [\mathbf{K}] = ([\mathbf{Y}]^{\mathrm{T}})^{-1}[\mathrm{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_n^2)][\mathbf{Y}]^{-1} \tag{2}$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are the modal frequencies of the model of dimension $n \times n$, and $[\mathbf{Y}] = [\{\mathbf{y}_1\}, \{\mathbf{y}_2\}, \ldots, \{\mathbf{y}_n\}]$ are the modal shapes of a non-damped structure. By using relations (2) desirable dynamic properties expressed in terms of known modal frequencies and modal shapes can be supplied to model (1).

We may assume that the presence of a small amount of the structural damping does not introduce any perceptible changes in the modal synthesis procedure. The proportional damping matrix is obtained as a linear combination of synthesized matrices [**M**] and [**K**] rather than by synthesizing matrix [**C**] directly. The coefficients α, β can be easily determined if the damping ratio values of the structure corresponding to two different modal frequencies are known. If damping forces are large, the full complex eigenvalue problem has to be treated and the modal synthesis performed by using complex mode shapes and frequencies.

In this work we neglect the damping forces completely by taking $\alpha = \beta = 0$.

3. 'OPTIMUM' COMPONENT SUBSTRUCTURES

In wave propagation models large parts of computational domains can be built of *component substructures* (CS). Figure 1(a) presents a CS containing 5×5 nodes(CS_5×5). A CS can be treated as a higher-order element, however, the principle of obtaining its matrices is different from traditional higher-order elements as the matrices of a CS are generated without directly employing the shape functions as interpolation tools. As a limiting case, a CS may consist of a single quadrilateral element, or may be a larger domain the shape of which is geometrically similar to the shape of the element. We need to optimally modify the spectral properties of a CS in order to produce the minimum modal frequency error of the whole structure.

In our approach, formation of the mode set for modal synthesis is performed as follows. Assume we need to synthesize the matrices of a CS the total number of degrees of freedom

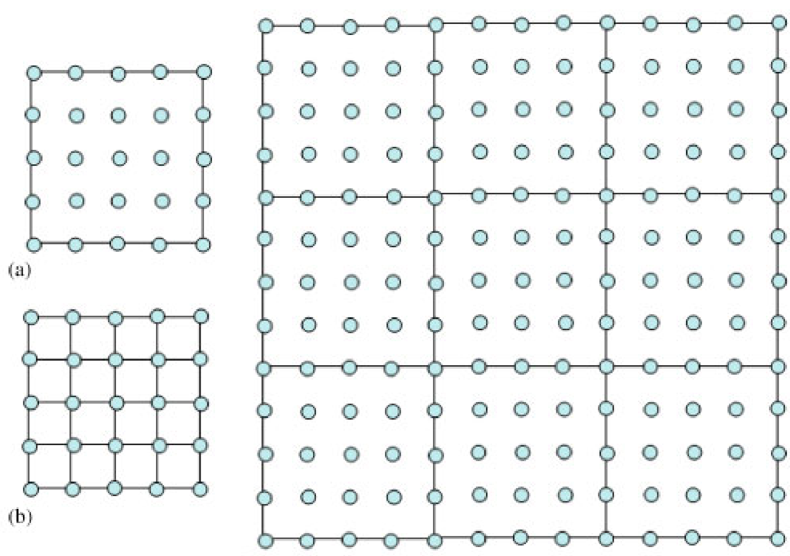

Figure 1. (a) Component substructure CS_5×5; (b) component substructure meshed by quadrilateral elements; and (c) quadrilateral sample domain 13×13 nodes assembled of nine CS_5×5.

(DOF) of which is n and the number of rigid body modes of the CS is r. Assume also that exact values of all n modal frequencies $0, 0, \ldots 0, \omega_{r+1}, \omega_{r+2}, \ldots, \omega_{r+n}$ of the CS are known (the first r frequencies are always zeros).

Though exact modal shapes of the CS may be known as well, we are not able to present them uniquely in the rough mesh. In Reference [8] we used a procedure based on projection of exact modal shapes on the rough mesh of a CS in 1D. In this work a slightly different approach has been used which worked more reliably in 2D case. We obtain the set of modal shapes used for modal synthesis by solving an eigenvalue problem for a free CS roughly meshed by using traditional elements (e.g. four node rectangles in 2D case) in such a way that all nodes of the CS are the nodes of the mesh, Figure 1(b). Moreover, for obtaining each jth modal shape a new jth eigenvalue problem is solved where generalized mass matrices of elements are employed as $[\mathbf{M}^e] = k_{\mathrm{L}j}[\mathbf{M}_{\mathrm{L}}^e] + (1 - k_{\mathrm{L}j})[\mathbf{M}_{\mathrm{C}}^e]$, $j = r+1, \ldots, n$. The weight coefficient $0 \leqslant k_{\mathrm{L}j} \leqslant 1$ specifies the contribution of the lumped mass matrix $[\mathbf{M}_{\mathrm{L}}^e]$ and $(1 - k_{\mathrm{L}j})$ correspondingly specifies the contribution of the consistent mass matrix $[\mathbf{M}_{\mathrm{C}}^e]$ to the generalized mass matrix $[\mathbf{M}^e]$ of an individual element. Only the jth modal shape is picked from the full set of modes presented by the solution of the jth eigenvalue problem and included into the modal shape set used later for modal synthesis. The values of coefficients $k_{\mathrm{L}j}$ used for obtaining each jth mode are not known in advance as finally they are obtained as a result of an optimization process described below. As an initial guess, values $k_{\mathrm{L}j} = 0.3 - 0.7$ could be regarded as a reasonable choice based on experience as for different physical environments and different types of elements the values of $k_{\mathrm{L}j}$ in this range ensure the best performance of the models meshed by traditional elements. In our numerical experiments we usually started form values $k_{\mathrm{L}j} = 0.5$. Finally the rigid body modes 1 to r are generated, and the mode set for modal synthesis reads as $\omega_1, \omega_2, \ldots, \omega_n$; $[\mathbf{Y}] = [\{\mathbf{y}_1\}, \{\mathbf{y}_2\}, \ldots, \{\mathbf{y}_n\}]$.

Modal frequencies and shapes are further modified by scaling them as

$$[\mathrm{diag}(0, \ldots, 0, \alpha_{r+1}^{\omega}\omega_{r+1}^2, \alpha_{r+2}^{\omega}\omega_{r+2}^2, \ldots, \alpha_{r+n}^{\omega}\omega_n^2)] = [\mathrm{diag}(\omega^2)]\{\boldsymbol{\alpha}^{\omega}\} \tag{3}$$

$$[\{\tilde{\mathbf{y}}_1\}, \ldots, \{\tilde{\mathbf{y}}_r\}, \alpha_{r+1}^{y}\{\tilde{\mathbf{y}}_{r+1}\}, \ldots, \alpha_n^{y}\{\tilde{\mathbf{y}}_n\}] = [\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^{y}\} \tag{4}$$

where $\{\boldsymbol{\alpha}^{\omega}\}^{\mathrm{T}} = \{1, \ldots, 1, \alpha_{r+1}^{\omega}, \ldots, \alpha_n^{\omega}\}$, $\{\boldsymbol{\alpha}^{y}\}^{\mathrm{T}} = \{1, \ldots, 1, \alpha_{r+1}^{y}, \ldots, \alpha_n^{y}\}$ are coefficients the values of which need to be specified. Initially, coefficients $\{\boldsymbol{\alpha}^{\omega}\}$ and $\{\boldsymbol{\alpha}^{y}\}$ have unity values.

The above presented modifications of the modal set preserve the physical essence of an unconstrained CS. The modal frequencies corresponding to the rigid body modes are zeroes and the modal shape vectors remain orthogonal and express essentially the same modal shapes as before the modification. Also the total mass of the CS remains unchanged.

The optimization is performed by assembling CS into *sample domains* of a shape similar to the shape of a CS provided that sufficiently large number of exact modal frequencies of the sample domain is known, Figure 1(c). As an example, for rectilinear and rectangular acoustic domains such modal frequencies are available analytically. In other cases a highly refined model of the sample domain can be used in order to obtain 'nearly exact' (say, $<0.5\%$ error) modal frequency values. Now the modal frequency error minimization problem can be formally presented as

$$\min_{k_{\mathrm{L}j}, \{\boldsymbol{\alpha}^{\omega}\}, \{\boldsymbol{\alpha}^{y}\}} \Psi = \sum_{i=r+1}^{\hat{N}} \left(\frac{\hat{\omega}_i - \hat{\omega}_{i0}}{\hat{\omega}_{i0}} \right)^2 \tag{5}$$

where the penalty-type target function presents the cumulative modal frequency error of the sample domain, $\hat{\omega}_i$ is the modal frequency of ith mode of the structure, $\hat{\omega}_{i0}$ is its exact value known theoretically or obtained by using a highly refined finite element model. Number $\hat{N}$ of modes contributions of modal errors of which are included into function (5) can be selected freely. It means that namely these $\hat{N}$ lower modes will have minimized modal errors. We suggest $\hat{N}$ to be taken as 30–80% of the total number of modes of the sample domain. Our experiments demonstrate that the percentage of minimized error (say, 0.5–2%) modes of the structure can be increased if larger individual CSs are employed. As an example, by using CS of size 2×2 (CS_2×2) modal errors of about 30% of modes of the domain can be expected to be made lower than 2%. Meanwhile, the CS_5×5 allow to achieve minimized modal errors over almost 80% of modes. The thorough discussion on this is presented in Section 4 of this paper.

Practically, the size of the sample domain is determined by a reasonable amount of calculations. Our numerical experiments demonstrate that often it is enough to perform optimization on a sample domain consisting of only several CSs, and the optimized matrices of a single CS work well if a considerably larger structure is assembled. We cannot present any theoretical proof of the validity of the approach, however, numerical experiments presented in Reference [8] and in this work illustrate that it works.

4. NUMERICAL RESULTS

Figure 2 presents the results obtained by investigating the modal properties of the quadrilateral acoustic sample domain 13×13 nodes. Figure 2(a) demonstrates the relative modal errors of the sample domain assembled of traditional quadrilateral elements and by using lumped, consistent and generalized mass matrices. The modal error distribution for the three types of models is quite typical. Lumped matrices have a tendency to diminish the modal frequency values. On the contrary, consistent mass matrices produce oversized modal frequency values. The errors of the models based on generalized mass matrices are always smaller, how-

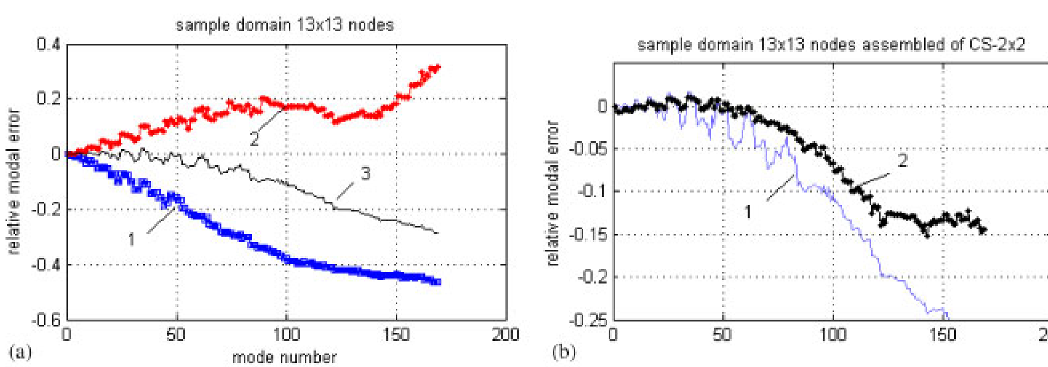

Figure 2. Relative modal errors of quadrilateral acoustic sample domain 13×13 nodes obtained by using: (a) quadrilateral elements with lumped (1), consistent (2) and generalized $k_{\mathrm{L}} = 0.35$ (3) mass matrices; and (b) CS_2×2 optimized over 30% of modes (2) compared to quadrilateral elements with generalized mass matrices (1).

ever the modal errors cannot be achieved to be close to zero over all modal frequency range. If we use the same generalized mass matrix for each individual element, optimum value of k_L may be easily found by a numerical experiment. The value $k_L = 0.35$ represented by curve 3 in Figure 2(a) produces nearly optimum modal frequency error for an acoustic domain assembled of traditional four-node quadrilateral elements. An analysis presented in Reference [8] for 1D case demonstrated that the 'minimum' cumulative modal frequency error over all modal frequency range is not necessarily the 'optimum'. The performance of wave propagation models was better if very small modal frequency errors over the range of 60–80% of lower modes could be ensured rather than an even distribution of the errors over all modal frequency range achieved.

Two curves are presented in Figure 2(b). The first one is the same as the 3rd curve in Figure 2(a). Here and further in the text it is used as a reference curve in order to compare and evaluate the performance of synthesized CS. The second curve in Figure 2(b) presents the results obtained by using optimized CS_2×2 as described in Section 3. The meshes used for obtaining both curves have been identical as geometrically CS_2×2 is also a four-node quadrilateral element. However, matrices of CS_2×2 are obtained by modal synthesis as a consequence of optimization process, therefore they are expected to produce lesser modal errors. In the numerical experiment number $\hat{N}$ in function (5) has been selected equal to $\sim$30% of the total number of modes of the sample domain. Consequently, in Figure 2(b), curve 2 approximately 30% of modes have modal errors not exceeding 2%.

Nevertheless, the optimized models assembled of CS_2×2 do not demonstrate a marked difference in the modal error distribution when compared to the generalized mass matrix models (curves 1 and 2 in Figure 2(b)). Much better results can be obtained by using larger CSs. CS_5×5 has been optimized to form structures with minimal modal errors over more than 80% of the total amount of modes of the structure. The optimization results are presented in Figure 3. Figure 3(a) presents the modal errors of the sample domain 13×13 nodes assembled of optimized CS_5×5. The same CS_5×5 assembled to 29×29 node domain give modal errors presented in Figure 3(b). In both figures curves displaying relative modal errors of the sample domain meshed by quadrilateral elements with generalized ($k_L = 0.35$) mass matrices are presented for the sake of comparison. In both cases all modal errors in the range of 80% of lower modes of the sample domain do not exceed 1–2% and are much lesser than can be achieved by employing traditional generalized mass matrices. On the other hand, Figure 3(a) and (b) justify the assumption that the modal error distribution over the modal frequency range is nearly independent upon the size of the sample domain.

The performance of the optimized CS with respect to traditional elements is demonstrated in Figure 4 by analysing the acoustic wave pulse propagating through a very roughly meshed domain. Excited by means of one sine pulse of normal velocity at the boundary excitation zone (Figure 4(a)) the circular wave front propagates and is reflected from the boundaries of the domain. The curves in Figure 4(b) demonstrate the propagating wave shapes in terms of the velocity potential obtained by using the optimized CS formulation and the reference wave shape obtained as a convergent solution of a densely meshed model. An excellent performance of the optimized model assembled of CS_5×5 is demonstrated where only $\sim$5 elements used per wave pulse length enabled to get the shape of the wave in close resemblance to the reference wave shape. Traditional quadrilateral elements used at such low mesh densities produce the resulting wave shape very different from the reference wave.

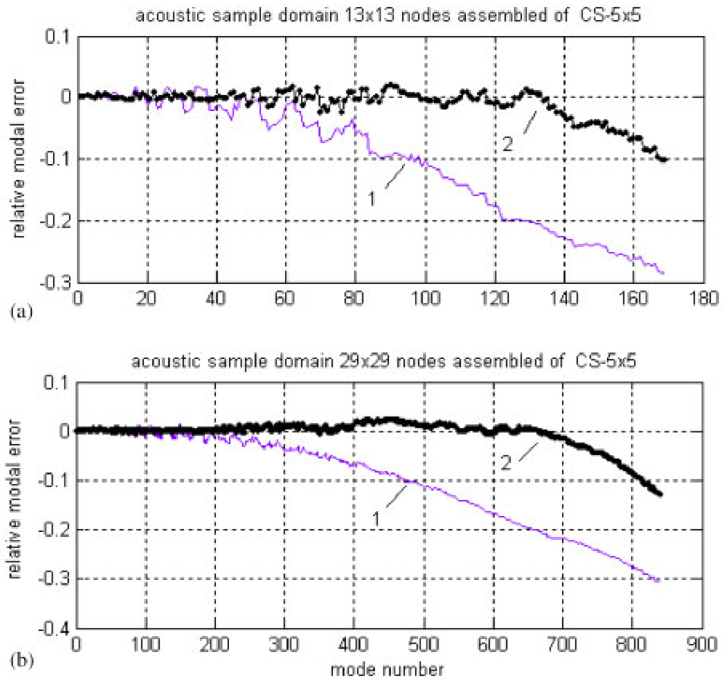

Figure 3. Relative modal errors of sample domains 13×13 (a) and 29×29 (b) nodes. 1—sample domain meshed by quadrilateral elements by using generalized ($k_L = 0.35$) mass matrices (presented here for the sake of comparison); and 2—sample domain assembled of CS_5×5 optimized over 80% of modes.

Similar results can be observed by analysing quadrilateral elastic domains. Figure 5 presents relative modal errors of lumped and consistent models. Generalized mass matrices obtained by using the weight coefficient $k_L = 0.35$ produce the relative error distribution less than 5% over all modal frequency range. Therefore, computational wave propagation models obtained by using such generalized mass matrices are expected to have a very good performance. By performing the optimization process of the CS_3×3 the relative modal errors can be further diminished, Figure 5(b). However, relative modal frequency errors of several modes could not be made lower than 4%.

5. CONCLUSIONS

The research presents highly convergent 2D computational models for wave propagation simulations consisting of rectangular substructures. The computational models are assembled of *optimized component substructures* obtained by performing the optimization of the modal properties of a sample domain. A CS can be treated as a higher-order element, however, the principle of obtaining its matrices is different from traditional higher-order elements as the matrices of a CS are generated without directly using the shape functions as interpolation

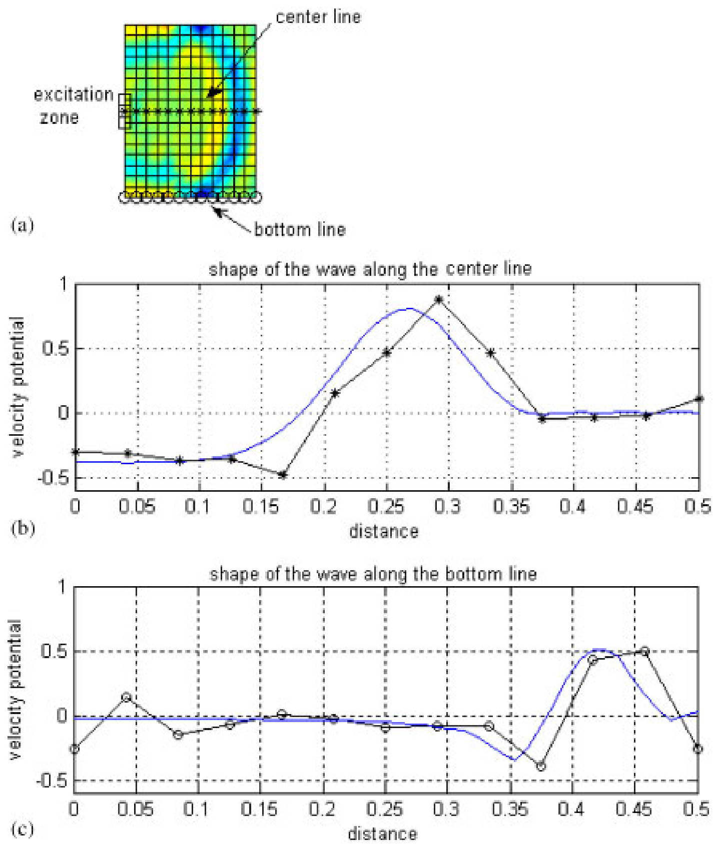

Figure 4. Acoustic wave propagating in a roughly meshed (13×17) domain: (a) rough mesh and velocity potential contour plot at a given time point; (b) reference shape of the wave (solid line) and the wave shape along the centreline of the model obtained by using the optimized CS_5×5 (marked line-$*$-); and (c) reference shape of the wave (solid line) and the wave shape along the bottom line of the model obtained by using the optimized CS_5×5 (marked line -o-).

tools. Optimized component substructures assembled to larger domains demonstrate the same modal error distribution over the modal frequency range as has been obtained for the sample domain. The same component substructures can be used for assembling real computational domains of any size.

Though the computation of CS matrices is a time consuming procedure, the amount of the computer resource necessary at this stage is not of a primary importance. As the obtained CS are used for solving linear problems of short wave propagation, their matrices do not need to be recalculated neither during the solution process nor before solving a new problem in a new domain.

When compared with lumped, consistent or generalized mass matrices, optimized component substructures produce significantly better results. However, the mass matrices of the optimized CS are non-diagonal. The obtained 2D models have very close-to-exact (less than 1–2% error)

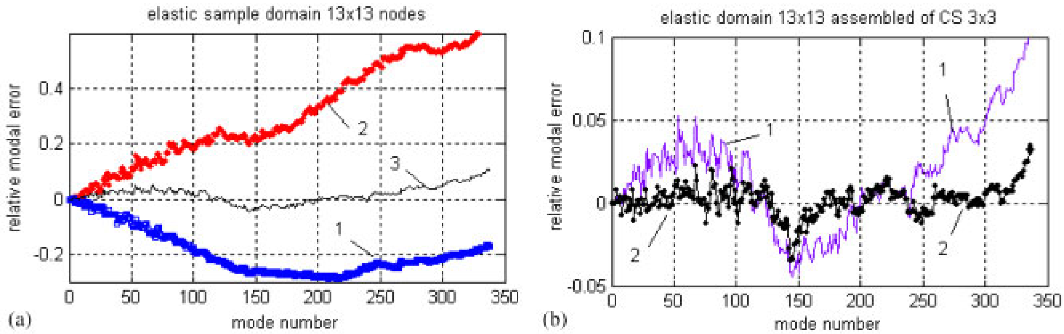

Figure 5. Relative modal errors of quadrilateral elastic sample domain 13×13 nodes: (a) obtained by using traditional elements with lumped (1), consistent (2) and generalized (3) mass matrices; and (b) obtained by using the CS_3×3 optimized over 80% of modes (2) compared to quadrilateral elements with generalized mass matrices (1).

modal frequency values of more than ∼80% of the total amount of modes of the structure and are able to present the propagating wave pulse shape by using only few nodal points per wavelength. The drawback is that at the moment we are not able to treat complex geometries by using the synthesized CS only. In complex geometries they should be combined with domains meshed by traditional elements.

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